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Energy level statistics in the transition region between integrability and chaos

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Abstract. A generic one-parameter family of billiards discovered and introduced by Robnik (1983) is used to study the spectral properties of corresponding quantum systems. When the parameter is varied a smooth transition from an integrable system over a typical KAM system to an almost ergodic system can be observed. We calculate up to 7600 lowest reliable energy levels.

A detailed analysis of the numerical data shows significant deviation from the semiclassical Berry–Robnik formulae for the nearest-neighbour level spacing distribution $P(S)$ except for large level spacings, $S > 1$, which can only be explained by a very slow convergence towards the semiclassical regime where these formulae are predicted to be correct. At small S the power-law level repulsion is clearly observed and a fit by the phenomenological formula by Izrailev is statistically significant.

1. Introduction

One of the fundamental questions in quantum chaos is the correspondence between the classical dynamics of a given Hamiltonian dynamical system and the statistical properties of the energy spectrum of its quantum counterpart (Bohigas and Giannoni 1984, Eckhardt 1988).

Classical Hamiltonian systems can exhibit three types of qualitatively different dynamical regimes. In the integrable case the stable classical motion in the $2f$-dimensional phase space (where f is the number of freedoms) is confined to f-dimensional invariant tori, which fill the entire phase space. The opposite extreme of ergodicity is characterized by a typically unstable, chaotic motion on the entire $(2f-1)$-dimensional energy surface where, by definition, almost every trajectory comes arbitrarily close to almost any other point on the energy surface. However, there is a mixed regime in the transition region between integrability and ergodicity where regions of stable regular motion on invariant tori (regular regions) coexist on the energy surface with the so-called irregular regions of chaotic motion (of presumably positive measure on which the motion is ergodic-like).

As for the statistical properties of quantum spectra of classically integrable and ergodic Hamiltonian systems, it has been shown that they represent two universality classes.

The energy spectrum of a classically integrable Hamiltonian system represents a sequence of completely uncorrelated numbers and the spectral fluctuations obey Poissonian statistics, as has been argued theoretically by Berry and Tabor (1977), and demonstrated numerically by several workers (see Casati *et al* 1985, 1986, Feingold 1985, Seligman and Verbaarschot 1986). In terms of the nearest level spacing distribution $P(S)$ ($P(S)\,\mathrm{d}S$ is the

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probability that energy spacing $E_{k+1} - E_k$ for some random index k lies between S and $S + \mathrm{d}S$), we have the result

$$P^P(S) = \mathrm{e}^{-S} \tag{1}$$

provided that the level sequence E_k is normalized to the unit mean level spacing, $\int \mathrm{d}S\, SP(S) = 1$.

On the other hand, in another extreme of ergodicity it has been conjectured by Bohigas *et al* (1984) that the statistical properties of quantum spectra can be modelled by the ensembles of random matrices, namely with a GOE (Gaussian orthogonal ensemble) in the case of time-reversal symmetry (or more generally, any other anti-unitary symmetry) or with GUE (Gaussian unitary ensemble) in the absence of such symmetry (Robnik and Berry 1986, Robnik 1986). There is growing numerical support for this conjecture, e.g. as given recently in Robnik (1992a, b) and in Bohigas *et al* (1990). Here we should also mention the numerically supported phenomenological theory of Wilkinson *et al* (1991) and Feingold *et al* (1991) which models the quantum Hamiltonian of a classically ergodic system in the basis of another ergodic Hamiltonian and makes use of the semiclassical estimates of the matrix elements (Feingold and Peres 1986, Feingold *et al* 1989). For theoretical arguments using the Gutzwiller approach see Wilkinson (1988). Their conclusion is again consistent with the conjecture of Bohigas *et al* (1984) except possibly in the case of two freedoms, where some doubts have been raised. In this universality class the energy levels are highly correlated, displaying *level repulsion*, which is very well approximated by the Wigner distribution for $P(S)$ (in the case in which there is anti-unitary symmetry)

$$P^W(S) = \tfrac{1}{2}\pi S \mathrm{e}^{-\pi S^2/4}. \tag{2}$$

It should be emphasized that this discovery of the universality classes of spectral statistics in the cases where the classical limit is integrable or ergodic is one of the important recent results in quantum chaos (see e.g. Gutzwiller 1990, Haake 1991). However, this classification scheme applies only to generic systems, and some notable exceptions are quite well known, for example the geodesic motion on the compact surfaces of constant negative curvature (Balazs and Voros 1986, Bohigas *et al* 1986, Aurich and Steiner 1989) and the billiards with a singular scatterer (Šeba 1990).

The intermediate regime of mixed dynamics in the classical phase space where regular and irregular regions coexist on the energy surface is also still a subject of intense research work. It has been shown for the first time numerically by Robnik (1984) that the transition between Poisson and GOE statistics is a continuous one, and this has been confirmed independently by Seligman *et al* (1984). Berry and Robnik (1984) elaborated a semiclassical theory for the energy level spacing distribution $P(S)$ in such a mixed regime. The theory rests upon an assumption about the semiclassical localization of eigenfunctions and the associated Wigner phase-space distributions either in classical regular or classical irregular regions. The sequences of levels associated with these regions are assumed to be statistically independent, and their mean spacing (or the level density) is determined by the invariant measure of the corresponding regions in classical phase space. The simplest two-component Berry–Robnik formula for one regular component with measure ρ_1 and one irregular component with measure $\rho_2 = 1 - \rho_1$ reads

$$P^{\mathrm{BR}}_{\rho_1}(S) = \{\rho_1^2 \mathrm{erfc}(\tfrac{1}{2}\sqrt{\pi}\rho_2 S) + (2\rho_1\rho_2 + \tfrac{1}{2}\pi\rho_2^3 S)\mathrm{e}^{-\pi\rho_2^2 S^2/4}\}\mathrm{e}^{-\rho_1 S}. \tag{3}$$

Initially there seemed to be some numerical evidence in support of the applicability of this semiclassical theory (e.g. Meyer *et al* 1984, Seligman *et al* 1985, Seligman and

Verbaarschot 1985), but the statistical significance was not good enough to draw firm conclusions. Moreover, recently there has been a number of numerical works (Wintgen and Friedrich 1987, Hönig and Wintgen 1989) showing that quite unexpectedly the Brody distribution (Brody 1973, Brody *et al* 1981) gives quite a satisfactory fit globally. The Brody distribution, which is given by

$$P_\beta^{\mathrm{B}}(S) = aS^\beta \exp(-bS^{\beta+1}) \qquad a = (\beta+1)b \qquad b = \{\Gamma((\beta+2)/(\beta+1))\}^{\beta+1} \tag{4}$$

is characterized by an important feature of power-law level repulsion at small S. It must be emphasized that this is only an *ad hoc* one-parameter family of distributions and in contrast to (3) has no deep physical background, but it interpolates between Poisson (1) ($\beta = 0$) and Wigner (2) ($\beta = 1$) in a simple way. Therefore it is frequently used but only as a mere reference function. The greatest discrepancy between the Berry–Robnik and Brody formulae is at small S. The former formula suggests constant behaviour $P_{\rho_1}^{\mathrm{BR}}(0) = 1-(1-\rho_1)^2 \neq 0$ at $S = 0$, while the Brody formula goes to zero as a power law, $P_\beta^{\mathrm{B}}(S \to 0) \propto S^\beta \to 0$. This phenomenological discrepancy can be explained by the fact that the Berry–Robnik formulae are not correct at small S for finite *non*-semiclassical spectra and, indeed, simple random-matrix-model considerations suggest that $P(S)$ should behave like $P(S \to 0) \propto S$ in the region whose width drops to zero as $\hbar$ goes to zero or, equivalently, as the number of levels increases (Robnik 1987). A physically more deeply founded version of the Brody-like distribution has been recently proposed by Izrailev (1988, 1989) as discussed in section 4 (see equations (31) and (32)).

The motivation for this work is to analyse this transition further and to provide additional numerical evidence in order to clarify the situation. The work is based on the method and approach developed in Robnik (1984). Thus we reconsider the generic one-parameter family of two-dimensional billiards studied there.

We have considerably improved the statistical significance of our numerical results, and we have observed significant deviation from the semiclassical Berry–Robnik formula except for large level spacings, $S > 1$. We have indeed observed power-law behaviour $P(S) \propto S^\beta$ at small S, $S < 1$, where the so-called 'level repulsion parameter' β smoothly varies from 0 to 1 as the transition parameter is varied. Globally we find statistically significant agreement with the phenomenological distribution of Izrailev (1988, 1989), cf Casati *et al* (1991) (which is reminiscent of Brody), but at large S the significance is comparable with that of the Berry–Robnik formula.

2. Geometry of the billiard and its dynamical properties

The domain of our billiards is given by the quadratic conformal map of the unit disk in a complex plane w,

$$\mathcal{B} = \{w | w = z + \lambda z^2,\ |z| \leqslant 1\}. \tag{5}$$

The particle moves freely inside the billiard and bounces off the boundary $\partial\mathcal{B}$ elastically. The classical and quantum mechanics of these systems have already been studied by Robnik (1983, 1984). See also Hayli *et al* (1987). It has been shown there that this one-parameter family of billiards exhibits a smooth transition from the integrable case (circle, $\lambda = 0$) to an almost ergodic case ($\frac{1}{4} \leqslant \lambda < \frac{1}{2}$). In order to obtain a smooth analytic boundary the parameter λ must lie in the interval $[0, \frac{1}{2})$ and figure 1 shows the shape of the billiard for

a few typical values of λ. For small values of the perturbation parameter λ the billiard is a typical KAM system whereas for larger values of λ only one chaotic region dominates the phase space with only a few stability islands covered with invariant tori. The total area in the bounce map (Poincaré surface of section) of these invariant tori becomes negligible when the shape of the billiard becomes non-convex, for $\lambda > \frac{1}{4}$.

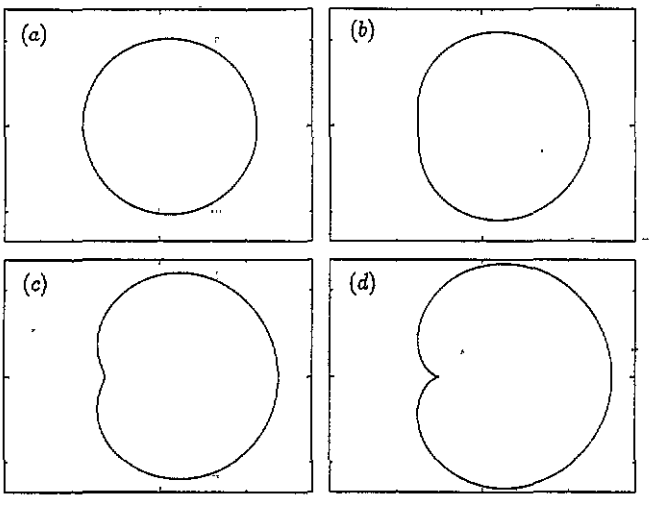

Figure 1. The shapes of the billiard boundary at four different values of the perturbation parameter λ: (*a*) 0.125, (*b*) 0.25, (*c*) 0.375 and (*d*) 0.5.

3. Quantization and the numerical method

Here we present a method for quantization of the billiard which turns out to be the most efficient numerically if one wishes to calculate all eigenenergies but no eigenvectors. The idea follows Robnik (1984) and is based on diagonalization in a truncated basis but it uses different ordering of the basis vectors in order to reveal a special and useful structure of the matrix.

We have to solve the following stationary Schrödinger equation

$$\Delta_w \psi(w) + (2m/\hbar^2) E \psi(w) = 0 \qquad \psi|_{\partial B} = 0 \tag{6}$$

where the index of the Laplacian denotes the variable with respect to which it acts. We can set the factor $2m/\hbar^2$ equal to 1 by choosing the proper energy units. Then we use the conformal mapping $w(z) = z + \lambda z^2$, and transform equation (6) to the unit disk where we introduce the polar coordinates (r, ϕ)

$$\Delta_z \psi'(r,\phi) + E(1 + 4\lambda^2 r^2 + 4\lambda r \cos(\phi))\psi'(r,\phi) = 0 \qquad \psi'(1,\phi) = 0 \tag{7}$$

where we use the conformal map of the Laplacian $\Delta_z = |\mathrm{d}w/\mathrm{d}z|^2 \Delta_w$ and denote the wavefunction (in the z-plane and polar coordinates) by $\psi'(r,\phi) = \psi(w(r\mathrm{e}^{\mathrm{i}\phi}))$. On the unit disk one explicitly knows the orthonormalized eigenfunctions of the $-\Delta_z$ operator, namely

$u_n(r,\phi)$ with eigenvalues z_n^2, thus $(\Delta_z + z_n^2)u_n = 0$, which can be used as a basis to express the wavefunctions in the form

$$\psi' = \sum_n (c_n/z_n)u_n. \tag{8}$$

When one inserts (8) into equation (7), multiply it with u_m and integrate over the unit disk one obtains

$$\sum_n (E^{-1}\delta_{nm} - J_{nm})c_n = 0 \tag{9}$$

$$J_{nm} = \frac{1}{z_n z_m}\int_0^1 \mathrm{d}r\, r \int_{-\pi}^{\pi} \mathrm{d}\phi (1 + 4\lambda^2 r^2 + 4\lambda r\cos(\phi))u_n u_m. \tag{10}$$

So the reciprocal energies can be calculated as the eigenvalues of the matrix J and the corresponding eigenvectors $\mathbf{c}$ can be used to determine the eigenfunctions according to the transformation formula (8). The problem is numerically well defined as soon as we have a finite truncated basis $\{u_n | n \in \mathcal{F}\}$ where $\mathcal{F}$ is a finite set of indices. The most convenient way to choose a finite basis from the infinite complete one is to pick all those basis functions u_n whose corresponding eigenvalue z_n is smaller than some prescribed threshold wavenumber k_{cutoff}, $\mathcal{F} = \{n | z_n < k_{\text{cutoff}}\}$. In other words, the finite sub-basis consists of all eigenfunctions up to a given energy (of the unperturbed system, i.e. circle).

Our billiard has one discrete geometric symmetry, namely the reflection symmetry with respect to the real axis, so we have two types of state; those with even parity $\psi(w) = \psi(w^*)$ and those with odd parity $\psi(w) = -\psi(w^*)$. These two types of state must be treated separately.

Now we can specify our basis functions; they are well known products of Bessel and trigonometric functions

$$u_{kl}(r,\phi) = R_{kl} J_k(z_{kl} r) \begin{cases} \cos(k\phi) & \text{even parity} \\ \sin(k\phi) & \text{odd parity} \end{cases} \tag{11}$$

with normalization constants

$$R_{kl} = \sqrt{2 - \delta_{k0}}/\sqrt{\pi} J_k'(z_{kl}) \tag{12}$$

where J_k' is the derivative of the kth Bessel function and z_{kl} is its lth zero. The matrix elements of J can be written as follows

$$\begin{aligned} J_{(kl),(k'l')} &= \langle u_{kl} | u_{k'l'} \rangle + 4\lambda^2 \langle u_{kl} | r^2 | u_{k'l'} \rangle + 4\lambda \langle u_{kl} | r\cos\phi | u_{k'l'} \rangle \\ &= \delta_{|k-k'|,0} \left\{ \delta_{ll'} + 4\pi\lambda^2 (1 + \delta_{k0}) R_{kl} R_{k'l'} \int_0^1 \mathrm{d}r\, r^3 J_k(z_{kl} r) J_{k'}(z_{k'l'} r) \right\} \\ &\quad + \delta_{|k-k'|,1} 2\pi\lambda (1 + \delta_{k0} + \delta_{k'0}) R_{kl} R_{k'l'} \int_0^1 \mathrm{d}r\, r^2 J_k(z_{kl} r) J_{k'}(z_{k'l'} r) \end{aligned} \tag{13}$$

from which a simple *selection rule* can be extracted

$$|k - k'| > 1 \Longrightarrow J_{(kl),(k'l')} = 0. \tag{14}$$

This selection rule suggests us how to order our truncated basis $\{u_{kl}|(k,l) \in \mathcal{F}\}$ to obtain the most convenient form of a matrix for numerical diagonalization. Let us denote the maximal number of radial nodes $K = \max\{k|(k,l) \in \mathcal{F}\}$ and maximal number of angular nodes at fixed number of radial nodes $B_k = \max\{l|(k,l) \in \mathcal{F}\}$. If we choose the ordering

$$(k,l): (1,1), (1,2) \ldots (1,B_1), (2,1), (2,2) \ldots (2,B_2) \ldots (K,B_K) \tag{15}$$

then the matrix will have 'block tri-diagonal' structure with blocks of variable sizes $B_1, B_2, \ldots, B_K$. Such a matrix can also be regarded as a symmetric banded matrix with a half-bandwidth equal to $B_1 + B_2$ because the inequalities $B_1 \geqslant B_2 \geqslant \ldots \geqslant B_K$ follow from the properties of zeros of Bessel functions. If one carefully looks at the table of zeros of Bessel functions one obtains an approximate formula $B_k \approx (K-k)/3$. If we denote the dimension of our basis by N we can further write $N = \sum_{k=1}^{K} B_k \approx K^2/6 \approx 3B_1^2/2$. We used the double precision NAG-library routine F01BWF (on the VAX 8800 computer) for calculating all eigenvalues of real banded symmetric matrix. The total number of operations for this algorithm is proportional to the square of dimension N times the bandwidth. In our case we have $\mathcal{O}(N^{2.5})$ because $B_1 = \mathcal{O}(\sqrt{N})$. This is a factor $\sqrt{N}$ faster than diagonalization of the full matrix (or the same matrix in a differently ordered basis) which is of the order $\mathcal{O}(N^3)$.

We must consider the fact that, since we used a finite truncated basis, only the low-lying levels are accurate. Experience shows that roughly 25% of levels are accurate within 0.01% of the average level spacing, 30% of levels within 0.1%, and 35% of levels within 1%. We have typically employed a basis of 12 000 basic functions for $\lambda = 0.050, 0.075, 0.175$ or 8200 basic functions for $\lambda = 0.100, 0.125, 0.150, 0.200, 0.225, 0.250, 0.375$. This enabled us to obtain 3800 or 2600 levels correspondingly, accurate within a few thousandths of the mean level spacing for each parity.

4. Numerical results

The most important and most commonly studied statistical measure of a given spectrum is the level spacing distribution $P(S)$. We would like to point out that in studying $P(S)$ it is not wise to analyse the numerical histogram (for $P(S)$) itself for two reasons. First, the bin size is always chosen in an arbitrary way and, second, by bining the data into bins we actually lose the information which we are working hard to compute. Instead we study the *cumulative level spacing distribution* $W(S) = \int_0^S \mathrm{d}s\, P(s)$ and some functions of it rather than the level spacing distribution $P(S)$ itself.

The first step in the numerical analysis of the spectrum is the so-called unfolding procedure (see e.g. Bohigas and Giannoni 1984), i.e. transforming the spectrum in such a way that the average level spacing in the neighbourhood of each level in the transformed spectrum is unity. If $\{E_n|n = 1, 2, 3 \ldots\}$ is our original computed energy spectrum then the most natural way to define the unfolding would be

$$e_n = N(E_n) \tag{16}$$

where $N(E)$ is a smooth part of the spectral staircase function, i.e. the average or expected number of levels with energy below E. In the case of smooth billiards, $N(E)$ can be calculated by the generalized Weyl asymptotic formula (Baltes and Hilf 1978)

$$N(E) = \frac{\mathcal{A}E}{4\pi} - \frac{\mathcal{L}\sqrt{E}}{4\pi} + \frac{1}{6} \tag{17}$$

where $\mathcal{A}$ is the area and $\mathcal{L}$ is the perimeter of the billiard. In our case, $\mathcal{A} = \pi(1+2\lambda^2)$, $\mathcal{L} = 4(1+2\lambda)E(\sqrt{8\lambda}/(1+2\lambda))$, where $E(k)$ is the complete elliptic integral of the second kind. We should consider levels with even and odd parity separately. For odd parity we can construct $N_{\text{odd}}(E)$ using the Weyl formula by considering only the upper half of our billiard with a border along the x-axis (because odd wavefunctions are zero there). Since we know that $N(E) = N_{\text{odd}} + N_{\text{even}}$ we can thus also calculate N_{even}. The exact results are

$$N_{\text{odd}}(E) = \frac{1+2\lambda^2}{8}E - \frac{(1+2\lambda)\mathrm{E}(\sqrt{8\lambda}/(1+2\lambda))+1}{2\pi}\sqrt{E} + \frac{5}{24} \tag{18}$$

$$N_{\text{even}}(E) = \frac{1+2\lambda^2}{8}E - \frac{(1+2\lambda)\mathrm{E}(\sqrt{8\lambda}/(1+2\lambda))-1}{2\pi}\sqrt{E} - \frac{1}{24}. \tag{19}$$

Now we consider the sequence of level spacings $S_n = e_{n+1} - e_n$. If there are N of them then we define the cumulative level spacing distribution simply as

$$W(S) = \frac{1}{N}\sum_{n=1}^{N}\theta(S - S_n) \tag{20}$$

where $\theta(x)$ is the Heaviside unit step function. In practice we order the set of level spacings, so that $S_1 \leqslant S_2 \leqslant \ldots \leqslant S_N$, and then we can write

$$W(S) = \frac{n}{N} \Longleftarrow S_n \leqslant S < S_{n+1} \qquad S_0 := 0 \qquad S_{N+1} := \infty. \tag{21}$$

Knowing that such a $W(S)$ is, in fact, a numerically measured quantity of some unknown exact distribution $\tilde{W}(S)$ we are motivated to estimate an expected statistical error of our 'measurement' denoted by $\delta W(S)$.

Assuming that ordered level spacings are mutually independent 'events' we define a probability that k of N randomly selected level spacings will be smaller than S, $R_k^N(S)$. According to the assumption $R_k^N(S)$ is a simple binomial distribution for fixed values of N and S, that is

$$R_k^N(S) = \binom{N}{k}(W(S))^k(1-W(S))^{N-k}. \tag{22}$$

The expected value of $\tilde{W}(S)$ is given by the share of the level spacings below S

$$\langle\tilde{W}(S)\rangle = \frac{\langle k\rangle}{N} = \frac{1}{N}\sum_{k=0}^{N}kR_k^N(S) = W(S). \tag{23}$$

Thus, the expected value of cumulative level spacing distribution $\tilde{W}(S)$ is the measured value $W(S)$. The statistical error (one sigma)

$$(\delta W(S))^2 = \left\langle\left(\frac{k}{N}\right)^2\right\rangle - \left\langle\frac{k}{N}\right\rangle^2 = \frac{1}{N^2}\sum_{k=0}^{N}k^2R_k^N(S) - (W(S))^2 \tag{24}$$

can also be calculated directly, giving the result

$$\delta W(S) = \sqrt{\frac{W(S)(1-W(S))}{N}}. \tag{25}$$

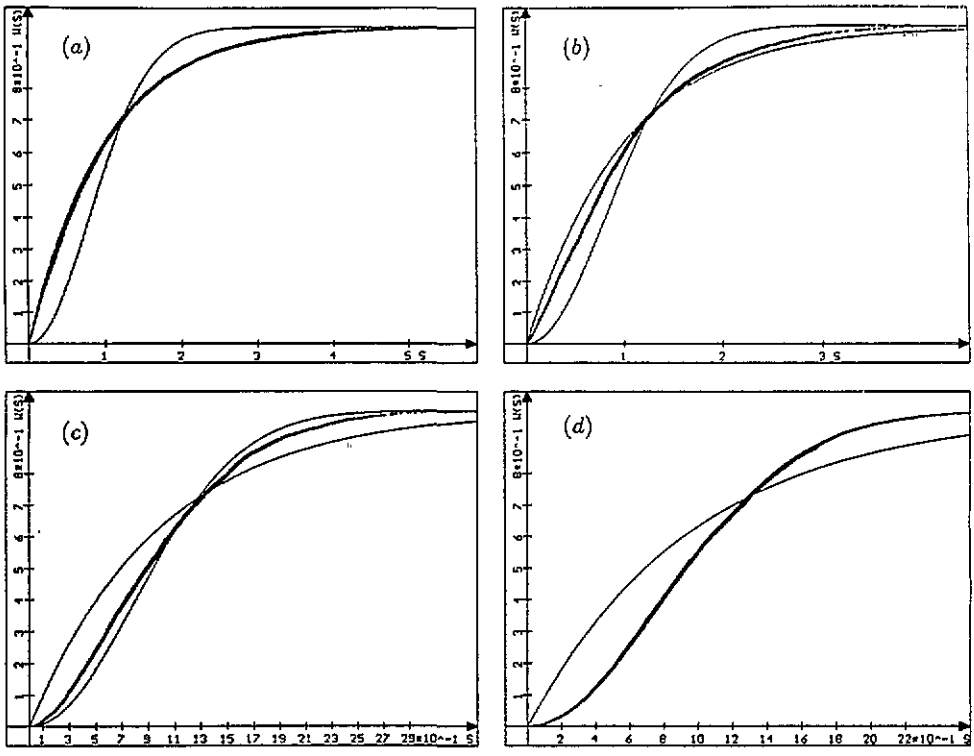

Figure 2. The cumulative level spacing distribution $W(S)$ at four different values of the perturbation parameter λ: (*a*) 0.05, (*b*) 0.15, (*c*) 0.2 and (*d*) 0.375. The gradual transition from Poisson towards Wigner-like distribution is clearly observed.

Figure 2 shows the cumulative distribution $W(S)$ for a few typical values of the parameter λ together with the exact Poisson and Wigner curves. The numerical $W(S)$ clearly shows a gradual transition from Poisson to Wigner type distribution as λ is varied from 0 (circle) to $\lambda \approx \frac{1}{2}$ (almost ergodic).

In order to clearly expose the qualitative features of $W(S)$ such as the power-law level repulsion at small S we will use another approach, using the analytically simple Brody distribution as the reference function. Analytic expressions for the Brody family of level spacing distributions (Poisson and Wigner are only special cases for $\beta = 0, 1$ respectively) are even simpler when written in cumulative form, namely

$$W_\beta^{\mathrm{B}}(S) = 1 - \exp(-bS^{\beta+1}) \tag{26}$$

where $b = b(\beta)$ was introduced in (4). One might think of the Brody family of distributions as the simplest one which interpolates between Poisson and Wigner distribution as the parameter β is varied form 0 to 1. The most interesting feature of the Brody distribution is the power-law behaviour at small S. One is tempted to test how well the real level spacing distribution can be fitted with the Brody distribution and, in particular, whether it behaves as a power law at small S. In order to study this, it is convenient to introduce the following functional transformation of the level spacing distribution, namely

$$T(\sigma) = \ln(-\ln(1 - W(\exp\sigma))). \tag{27}$$

This unusual transformation has the property that it transforms the Brody distribution to the straight line

$$T_\beta^{\mathrm{B}}(\sigma) = (\beta + 1)\sigma + \ln b. \tag{28}$$

Using this representation it is very easy to test by linear regression or even by the naked eye whether and how well the straight line can fit our numerical data. If so, then the slope of the fitting line minus one gives the so-called *level repulsion parameter* β. One can use equations (25) and (27) to calculate the expected error of the T-function

$$\delta T(\sigma) = \frac{\sqrt{W(\exp\sigma)}}{\sqrt{N(1 - W(\exp\sigma))|\ln(1 - W(\exp\sigma))|}} \tag{29}$$

which unfortunately diverges at $\sigma = \pm\infty$. This implies that the T-function is quite uncertain at very small and very large level spacings. Otherwise the T-function turns out to be very useful and we can now demonstrate the power-law behaviour over a few orders of magnitude (for S from a few times 10^{-3} up to about 1) (see figure 3). As can be seen, only at large level spacings ($S > 1$) do the deviations from the straight line become significant. It can be concluded that at large S the real $P(S)$ decays faster than the best fitting Brody distribution.

Although the T-function can help us to demonstrate power-law behaviour it is inconvenient in studying the global properties of the level spacing distribution. There are two reasons for this. First, the estimated error function δT is non-uniform in the sense that it gives different weights to different 'experimental points' $\sigma_n = \ln S_n$. Second, the numerical points are distributed non-uniformly over the σ-axis, in particular they are denser at larger σ. This means that the right-hand side of the figure carries more information than the left-hand side. We would like to plot our data in such a way that each unit area of the graph carries an equal amount of information. First we introduce a transformation $U(W)$

$$U(W) = (2/\pi)\cos^{-1}\sqrt{1 - W} \tag{30}$$

which has the property that the estimated error for $U(W(S))$ is a constant—equal for all level spacings S, $\delta U(S) = |\mathrm{d}U/\mathrm{d}W|\delta W(S) = 1/(\pi\sqrt{N})$. Now we apply the best fitting Brody distribution $W_\beta^{\mathrm{B}}(S)$ to our spectrum and plot the deviation $U(W(S)) - U(W_\beta^{\mathrm{B}}(S))$ against $W_\beta^{\mathrm{B}}(S)$. Since we know that $W_\beta^{\mathrm{B}}(S)$ gives quite a good global fit the abscissae of 'experimental points' $W_\beta(S_n)$ will be fairly equidistant, i.e. uniformly distributed. Such graphs show the fine-scale deviations from the Brody distribution (see figure 4) which now become significant.

There is another one-parameter family of level spacing distributions which interpolates between the Poisson and Wigner distribution devised by Izrailev (1988, 1989) on phenomenological grounds. This distribution stems from the study of nearest-neighbour eigenangle spacing distribution for the kicked rotator. It displays a smooth transition from Poissonian statistics for small values of the parameter, namely the kick strength k, to the GOE (Wigner type) for large values of k, although the classical dynamics is fully chaotic in all cases. This effect is a consequence of the interplay between the diffusion in phase space and the quantum localization and is typical of time-dependent (driven) systems. Izrailev distribution could also be relevant to our problem, because in fact he used only the idea of generalizing the joint distribution of eigenangles $P(\{\phi_i\}) \propto \prod_{n\neq m} |\mathrm{e}^{\mathrm{i}\phi_n} - \mathrm{e}^{\mathrm{i}\phi_m}|^\beta$ to all values of β (not only for $\beta = 0$ (Poisson), 1 (GOE), 2 (GUE)). This reasoning so far has only phenomenological support. Izrailev suggests the formula (see also Casati *et al* (1991))

$$P_\beta^{\mathrm{I}}(S) = AS^\beta(1 + BS)^{f(\beta)}\exp\{-\tfrac{1}{16}\beta\pi^2S^2 - \tfrac{1}{2}\pi(1 - \tfrac{1}{2}\beta)S\} \tag{31}$$

$$f(\beta) = (2^\beta(1 - \beta/2))/\beta - 0.168\,74 \tag{32}$$

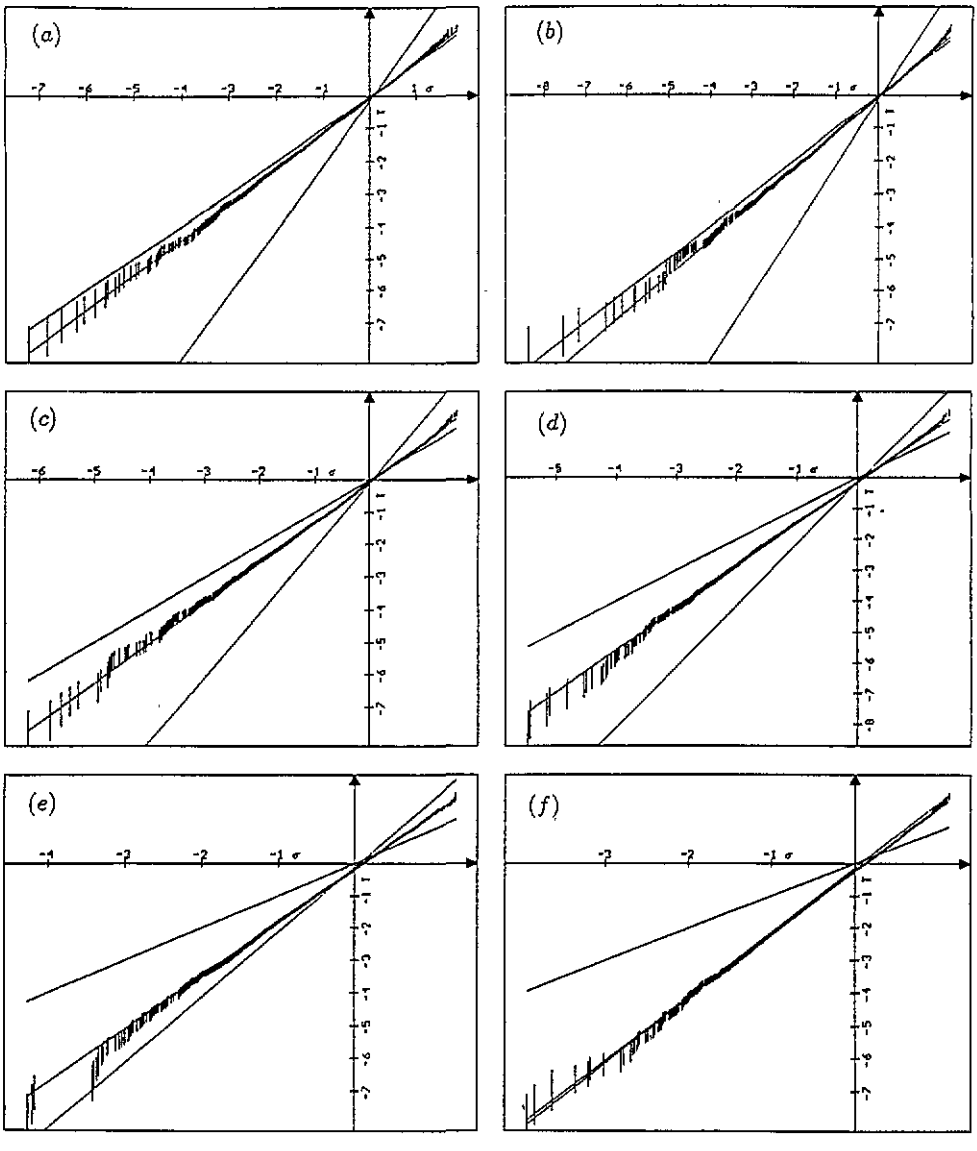

Figure 3. T-function against σ at six different values of the perturbation parameter λ: (*a*) 0.1, (*b*) 0.125, (*c*) 0.15, (*d*) 0.175, (*e*) 0.2 and (*f*) 0.375. The two straight lines corresponding to the Poisson and Wigner distributions are shown for comparison. The region of power-law level repulsion is clearly extended over several orders of magnitude and with varying parameter λ clearly shows a continuous transition from Poisson to Wigner. For the corresponding values of the level repulsion parameter β see table 1. The size of the bars indicates the estimated $\pm$one-sigma error.

where the parameters A and B are determined by the normalization conditions $\int \mathrm{d}S\, P_\beta^{\mathrm{I}}(S) = 1$, $\int \mathrm{d}S\, S P_\beta^{\mathrm{I}}(S) = 1$. We have tested this formula and found a significant match with numerical data, much better than for the Brody distribution (see figure 4 and table 1). The larger values of the level repulsion parameter β as determined by the Brody fit, as compared with those determined by the Izrailev fit, can be understood as follows. The domain of the

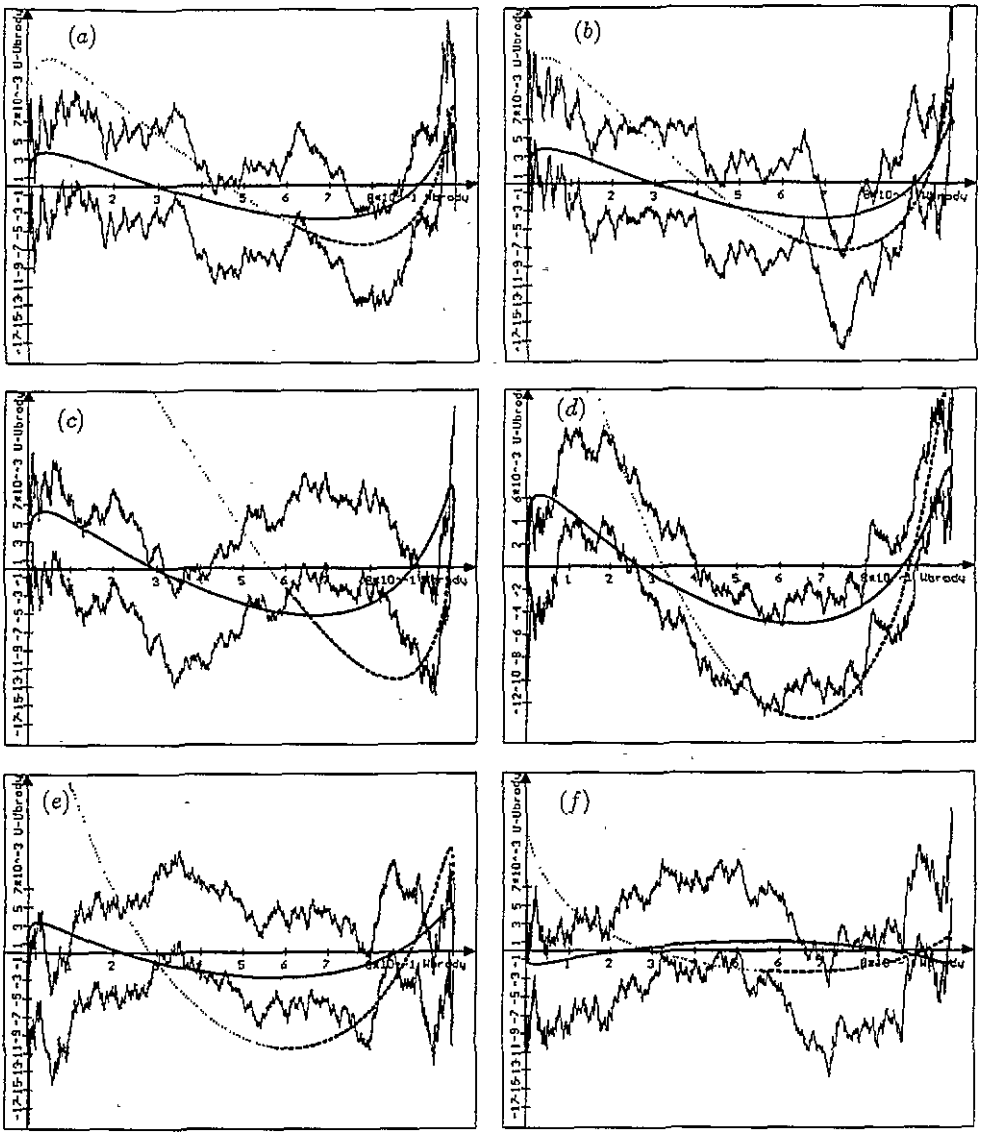

Figure 4. Fine-scale representation of the deviation of the numerical level spacing distribution $P(S)$ from the best fitting Brody distribution. We plot $U(W(S)) - U(W_\beta^B(S))$ against $W_\beta^B(S)$. The upper and the lower noisy curves represent one-sigma deviation from the actually calculated numerical data which thus lie in the middle of the band. The one-sigma error is constant along the abscissa in this representation. The smooth curve represents the best fitting Izrailev distribution while the broken curve represents the best fitting Berry–Robnik distribution based on the data of the dashed region. The dotted region is just an extrapolation of the obtained distribution. The chosen values of the parameter λ in (a) to (f) are the same as in figure 3. The determined values of the parameters of the best fitting distributions are given in table 1.

greatest deviation from the Brody distribution is in the asymptotic tail ($S \to \infty$) where the slope of the numerical data increases in the T-function representation which leads to a larger slope in the global straight-line fit i.e. β increases.

Table 1. The parameters and the chi-squares divided by the number of 'experimental points' of the best fit to the Brody, Izrailev and Berry–Robnik formulae for level spacing distribution. In the last column an accurate value of ρ_1 is given as determined by classical dynamics. The inequality signs indicate that a given quantity has definitely not yet converged and it changes in the indicated direction as the number of levels in the spectral sample increases (towards the semiclassical limit). The values in the table were determined using 7600 levels of both parities for $\lambda = 0.050, 0.075, 0.175$ and 5200 levels for all other values of λ. In all cases where convergence is clearly not yet observed the lower half of the spectral sample was not included in the statistics (since they would spoil the semiclassical limiting behaviour), whilst in all other cases 1200 lowest levels were discarded for the same reason.

λ	β_B	χ^2/N	β_I	χ^2/N	ρ_1^q	χ^2/N	ρ_1^{cl}
0.050	<0.04	1.0	<0.03	0.7	>0.76	0.2	0.98
0.075	<0.07	0.6	<0.05	0.3	>0.74	0.2	0.96
0.100	0.10	0.7	0.07	0.3	>0.68	0.2	0.88
0.125	0.10	1.0	0.08	0.4	>0.65	0.2	0.70
0.150	0.24	0.6	0.19	1.3	0.54	1.8	0.36
0.175	0.37	1.1	0.31	0.4	0.30	0.3	0.17
0.200	0.64	0.4	0.59	0.8	0.18	1.6	0.05
0.225	0.83	0.4	0.82	0.4	0.07	0.4	~0
0.250	0.92	0.4	0.95	0.5	0.03	0.2	~0
0.375	0.94	0.4	0.97	0.4	0.02	0.3	~0

We have also used the Berry–Robnik formula for the case of only one stochastic (irregular) component and one integrable (regular) component of phase space since the examination of the classical phase space (Poincaré surface of section) shows that only one chaotic region dominates (the SOS) at all values of λ, except for very small λ where all chaotic regions can be neglected. The global fit by the Berry–Robnik formula is not meaningful because of the large deviations at small S. Nevertheless, when only the data for sufficiently large level spacings ($S > 1$) are used for this purpose then the fit becomes statistically significant. In this way the measure ρ_1 of the regular component in the phase space can be estimated and compared with the more accurate result obtained by studying classical mechanics (see table 1). The two quantities, the quantal ρ_1^q and the classical ρ_1^{cl}, actually disagree and this problem will be discussed in the next section.

We have also studied the spectral rigidity defined as

$$\Delta(L) = \left\langle \min_{a,b} \frac{1}{L} \int_{e-L/2}^{e+L/2} \mathrm{d}e' \{n(e') - ae' - b\}^2 \right\rangle \tag{33}$$

$$n(e) = \sum_k \theta(e - e_k) \tag{34}$$

where the angular brackets in (33) indicate averaging over the spectrum (over the variable e). The results for a several typical values of the parameter λ are shown in figure 5. They qualitatively interpolate between the Poisson and GOE regimes. It is hoped that we can get some independent information about the parameter ρ_1 by the best fit to the simplest semiclassical formula for the spectral rigidity (cf Seligman and Verbaarschot 1985)

$$\Delta(L) = \Delta_{\mathrm{Poisson}}(\rho_1 L) + \Delta_{\mathrm{GOE}}((1 - \rho_1)L). \tag{35}$$

Because of the saturation effect (see figure 5) (which is a consequence of the finite spectrum) we are limited to a relatively small region $L < 10$. According to Berry's (1985) semiclassical theory of spectral rigidity the saturation effects set in at $L > L_{\max} \approx$

$2\sqrt{\pi \mathcal{A} N_{max}}/l_0$ where l_0 is the geometrical length of the shortest periodic orbit (twice the length of the shortest diameter; we have $l_0 = 4$), $\mathcal{A}$ is the area of the billiard and N_{max} is the sequential number of the highest energy level included in the spectral sample. In our case we have the estimate $L_{max} \approx 60$. In fact, the saturation typically starts at somewhat smaller values of L (in our case about $L \approx 12$, see figure 5) (cf Robnik 1992a).

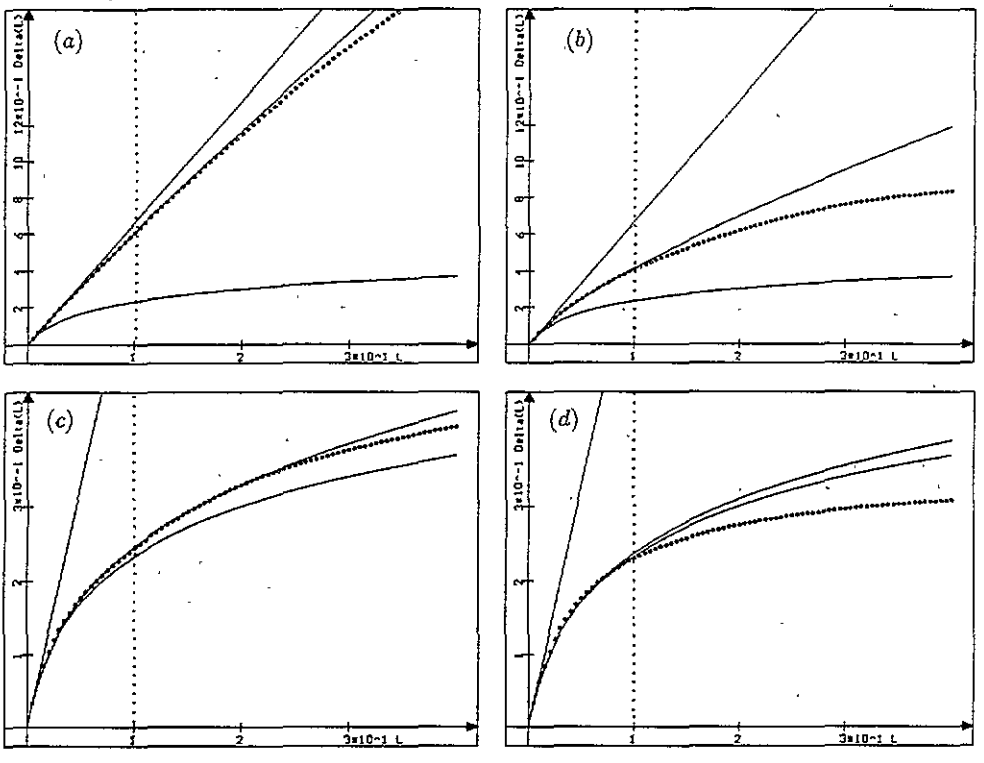

Figure 5. The spectral rigidity as described by $\Delta(L)$. The Poisson line and GOE curve are shown in full. The numerical data (circles) are fitted in the range $0 \leqslant L \leqslant 10$ by the semiclassical formula (35) (full curve). The gradual transition from Poisson to GOE is seen as the parameter λ is varied, λ: (*a*) 0.05, (*b*) 0.175, (*c*) 0.25 and (*d*) 0.375. As can be seen the saturation effects start at $L \approx 12$ which is notably smaller than the theoretical estimate $L_{max} \approx 60$ (see text).

Although the fit to the formula (35) is not very sensitive to small variations in the parameter ρ_1 in this region of L, we have determined ρ_1 by the best fitting curve (35). In table 2 we give the resulting values of ρ_1 at various values of λ and see that they agree very well with the values read from the best fit by the Berry–Robnik semiclassical formula for $P(S)$ (see table 1), but disagree with the accurate values determined from the classical dynamics.

5. Related theoretical developments and possible interpretations

The question which now arises is how to interpret our numerical results. We have found significant deviations from the semiclassical formulae although we considered finite but large spectral samples with up to almost the 8000th energy level of our Hamiltonian system which we believe is generic. There are two possible answers: either the basic assumption of

Table 2. The values of the quantal parameter ρ_1 for a few values of λ which are determined by fitting the spectral rigidity $\Delta(L)$ with the semiclassical formula (35) in the region $0 \leqslant L \leqslant 10$. The inequality signs in the table indicate that a given quantity has definitely not yet converged for the finite spectral sample (of 2600 to 3800 levels of fixed parity) but it still changes in the indicated direction as the semiclassical limit is approached. The spectral samples are the same as in table 1.

λ	ρ_1^{even}	ρ_1^{odd}
0.050	>0.76	>0.77
0.075	>0.77	>0.72
0.100	0.67	0.66
0.125	0.70	0.63
0.150	0.50	0.50
0.175	0.31	0.29
0.200	0.18	0.17
0.225	0.05	0.04
0.250	0.02	0.02
0.375	0.00	0.02

statistical independence in the derivation of the semiclassical formulae of Berry and Robnik (1984) is not satisfied or the semiclassical limit has not yet been established.

The basic assumption in the derivation of the semiclassical formulae rests upon the idea that the Wigner phase-space distribution of each eigenstate always condenses on a single classically ergodic component (either connected ergodic-like region or invariant torus) in the semiclassical limit. This assumption has so far been rigorously proven only for the special case of integrable systems (Berry 1977a). In another extreme of ergodicity this assumption implies that the Wigner phase-space distribution of each eigenstate is (almost) microcanonical (Voros 1976, Berry 1977b), except for the scars (Heller 1984, Berry 1989, Robnik 1992c). We should mention our preliminary results of a separate work in which we numerically study the projections of the Wigner phase-space distributions to the Poincaré surface of section for the same family of billiards in the mixed regime. So far we have found nothing which would disprove the basic assumption (Prosen and Robnik 1992).

It should be mentioned that Bohigas *et al* (1990) have recently analysed in detail a 2D system of two coupled 1D quartic oscillators and numerically verified the validity of the Berry–Robnik surmise in the semiclassical limit. Using the presence of dynamical quasi-degeneracy of regular energy levels (due to discrete symmetries in their system) they were able to separate the regular and irregular levels clearly. The irregular levels were shown to be associated with one chaotic component which, however, possesses a finite-time structure due to partial barriers in classical phase space. They have successfully explained the deviations in the number statistics of the irregular spectral component from the GOE as being due to badly articulated ergodicity, i.e. due to slow transitions between the five chaotic basins. But their results clearly show the approach to GOE in the semiclassical limit $\hbar \to 0$, as the number of energy levels in the spectral sample goes to infinity. This is yet another confirmation of the Bohigas *et al* (1984) conjecture. But it should be emphasized that their system is already quite far from the regime of soft KAM chaos.

So, there remains the second possibility of very slow establishment of the semiclassical limit, which seems more plausible. Even though the semiclassical formulae (for $P(S)$ and $\Delta(L)$) give the same but incorrect results† for ρ_1, and completely incorrect $P(S)$ for small

† That is ρ_1, as determined by the best semiclassical fit to $P(S)$ by the Berry–Robnik formula, is in excellent agreement with the value of ρ_1 as determined by the best semiclassical fit to $\Delta(L)$, but disagrees with the classical

S, this can be qualitatively understood. Small level spacings can feel the correlations among level sequences which correspond to Wigner phase-space distributions (of eigenstates) with semiclassically disjoint supports. This effect was expected (Robnik 1987) and it is only surprising that it is so large. On the other hand, we believe that the incorrect values of ρ_1 could be corrected if much larger portions of the spectrum would be calculated to and included in the evaluation of the statistics. For small values of the perturbation parameter λ the chaotic regions in phase space are small (they form a fractal set) and that may be the reason why the semiclassical limit is still being slowly formed so that the values of β and ρ_1 are clearly seen to change when the spectral sample increases. At larger values of λ ($\lambda > 0.1$) such a change has not been observed, perhaps because it is even slower, since the fine-scale details of the chaotic regions in the classical phase space are only gradually resolved by quantum waves as their wavelength goes to zero. If this qualitative interpretation is correct then we ought to understand theoretically (also in a quantitative manner) why the convergence is so slow, which at present is not possible.

We conclude that here we have dealt with too few levels to be able to study the semiclassical properties of the spectrum. One can see that in this mixed regime the global fit is described quite well by the phenomenological Izrailev distribution. In order to explain this observation we studied in a separate work (Prosen and Robnik 1993) the properties of the random matrix ensemble which models dynamical systems in the transition region between integrability and chaos (which become KAM systems when they come sufficiently close to the integrability). This ensemble is expected to capture all the average statistical correlations between the energy levels, but completely ignores all dynamical correlations at the outset. The idea for constructing this ensemble comes from the classical perturbation theory, where one writes the KAM Hamiltonian as a sum of an integrable part in terms of canonical actions plus small perturbation which can be expanded into harmonic (Fourier) series in canonical angles. We consider the semiclassical eigenfunctions (as determined by the tori quantization) of the integrable part as a basis of the Hilbert space. If this basis is ordered according to the increasing eigenvalues of the integrable part then the Hamiltonian becomes a sparsed banded matrix. We assume random distribution of the positions of non-zero offdiagonal elements and of their values and define a sparsed banded random matrix ensemble (SBRME). This random matrix ensemble is described by three parameters: the average increment α of Poissonian distributed diagonal elements, the bandwidth b which dictates $|i-j| > b \Rightarrow H_{ij} = 0$, and the average number m of uniformly distributed off-diagonal non-zero elements in each row. (The latter are assumed to be Gaussian random variables with zero mean and unit second moment.) SBRME is a generalization of the BRME as defined by Wilkinson *et al* (1991) where $m = b$. We have found that for SBRME the level spacing distribution $P(S)$ can again be quite well fitted by the Brody distribution, especially for sufficiently small ratio m/b, the sparsity. The latter becomes small in the semiclassical limit because the scaling laws with $\hbar$ and the perturbation parameter ϵ are $\alpha = \mathcal{O}(\hbar^f/\epsilon)$, $b = \mathcal{O}(\hbar^{1-f})$, $m = \mathcal{O}(1)$. The preliminary results of this research in progress show that the qualitative behaviour of the level repulsion parameter β as the function of ϵ is correct ($\beta(\epsilon)$ is a monotonically increasing function) which is not the case if the BRME ($m := b$) is used. Of course, we numerically observe the transition between these two regimes as the sparsity is varied.

We are thus faced with the not yet solved paradox that for almost any finite spectrum some kind of Brody-like distribution for $P(S)$ seems to be adequate (in particular at small S) even if we are confident that the semiclassical Berry–Robnik formula should hold for the

ρ_1. See tables 1 and 2.

infinite spectrum. We should be aware of the fact that we cannot reproduce semiclassical formulae using any random matrix ensemble since we lose all dynamical correlations by the assumptions of randomness in the very definition of the ensemble. We speculate that these so-called dynamical correlations may not be so important for finite but quite large spectra (and that may be the reason why random matrix theories work so well) but they must become crucial for infinite (semiclassical) spectra. We can only conclude that such a semiclassical transition apparently occurs quite late (a typical threshold size of a spectrum must be larger than 10^4 at least in the case of our Hamiltonian) and it can hardly be studied with present computer capabilities.

6. Discussion and conclusions

In this paper we have presented the revised and statistically significantly improved results on the energy level statistics of the billiard Hamiltonian introduced and studied by Robnik (1983, 1984). We have analysed the level spacing distribution $P(S)$ and the spectral rigidity $\Delta(L)$ in the transition region between integrability and chaos. At small S, such as $0 < S < 1$, we clearly find the power-law level repulsion and a good fit by the Brody distribution and, more importantly, an even better fit by the phenomenological formula devised by Izrailev (1988, 1989). See also Casati *et al* (1991). In this small-S region the semiclassical formula by Berry and Robnik (1984) is found to be inadequate. This has been qualitatively predicted (Robnik 1987), except that now this region of discrepancy is larger than expected and it displays a general power law rather than linear level repulsion. Namely, the level repulsion parameter β varies continuously with the deformation parameter between 0 (Poisson, circle) and 1 (Wigner-like, ergodic-like). However, for larger S, say $S > 1$, we do find a statistically significant fit by the Berry–Robnik formula for $P(S)$ (of approximately the same quality as the formula by Izrailev), except that the quantal parameter ρ_1 deviates from the invariant measure of the regular component as determined by the classical dynamics. The quantal parameter ρ_1 as determined by the semiclassical formula for the spectral rigidity $\Delta(L)$, for $0 \leqslant L \leqslant 10$, however, agrees very well with the quantal ρ_1 based on $P(S)$. We conclude that the semiclassical formulae (for $P(S)$ and $\Delta(L)$) are correct, but that the semiclassical limit is approached very slowly. The deviation of the quantal ρ_1 from the classical ρ_1 is explained qualitatively by correlations between the nearby energy levels due to the partial overlap of the corresponding Wigner phase-space distributions of eigenstates which are not yet sufficiently localized on their semiclassical supports. We propose that the small-S region shrinks as we approach the semiclassical limit, and that in this region all the dynamical correlations are typically averaged away and are therefore not important. Thus we propose, as explained in detail in a separate work (Prosen and Robnik 1993), that this somewhat universal behaviour of $P(S)$ at small S for finite spectra can be correctly captured by the sparsed banded random matrix ensemble (SBRME).

In another separate work (Prosen and Robnik 1992) we analyse the numerical Wigner phase-space distributions of eigenstates in the mixed regime and discuss the validity of the basic assumption upon which the semiclassical theory by Berry and Robnik (1984) rests. In this semiclassical limit we expect the dynamical correlations to become important, the SBRME to becomes inadequate and the semiclassical formulae of Berry and Robnik to apply.

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